Fie $f:(1,\infty)\rightarrow \mathbb{R}, f(y)=\int_0^y \frac{1}{x^2-2x+y}\,dy$. Calculați $\int_2^{10} f(y)\,dy$.

Soluție:

Pas 1 : Căutăm să formăm "pătrat perfect" la numitor.

$$f(y)=\int_0^y \frac{1}{x^2-2x+y}\,dy=\int_0^y \frac{1}{x^2-2x+1+y-1}\,dy=\int_0^y \frac{1}{(x-1)^2+\left(\sqrt{y-1}\right)^2}\,dy$$

$$=\frac{1}{\sqrt{y-1}}arctg\left(\frac{x-1}{\sqrt{y-1}}\right)\Bigg|_0^y=\frac{1}{\sqrt{y-1}}\left(arctg\left(\frac{y-1}{\sqrt{y-1}}\right)-arctg\left(\frac{-1}{\sqrt{y-1}}\right)\right)$$

Observație: Cum funcția arctg este o funcție impară ($arctg(-x)=-arctg(x)$), vom avea:

$$arctg\left(\frac{-1}{\sqrt{y-1}}\right)=-arctg\left(\frac{1}{\sqrt{y-1}}\right)$$

$$\Rightarrow f(y)=\frac{1}{\sqrt{y-1}}\left(arctg\left(\sqrt{y-1}\right)+arctg\left(\frac{1}{\sqrt{y-1}}\right)\right)$$

Observație:

$$\boldsymbol{arctg(x)+arctg\left(\frac{1}{x}\right)=\frac{\pi}{2}}$$

$$\Rightarrow f(y)=\frac{\pi}{2\sqrt{y-1}}$$

Pas 2 :

$$\int_2^{10} f(y)dy=\frac{\pi}{2}\int_2^{10}\frac{1}{\sqrt{y-1}}dy=\frac{\pi}{2}\left(2\sqrt{y-1}\right)\Big|_0^y=\frac{\pi}{2}\left(2\sqrt{9}-2\sqrt{1}\right)=2\pi\,.$$

Răileanu Dragoș - Iulian